

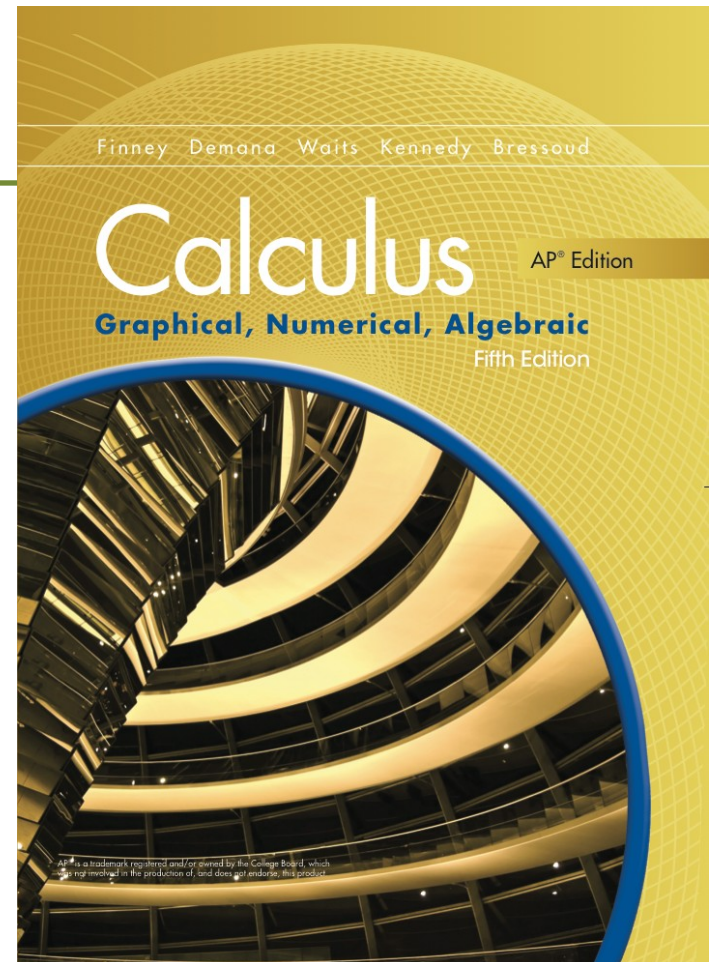
Chapter 5

Applications of Derivatives



Section 5.3

Connecting f' and f''
with the graph of f



Quick Review

Factor the expression and use sign charts to solve the inequality.

1. $x^2 - 4 < 0$

2. $x^3 - 9x > 0$

Find the domain of f and f' .

3. $f(x) = 2xe^x$

4. $f(x) = x^{1/3}$

5. $f(x) = \frac{x}{x-1}$

Quick Review

Find the horizontal asymptotes of the function's graph.

6. $f(x) = (9 - x^2) e^x$

7. $f(x) = (9 - x^2) e^{-x}$

8. $f(x) = \frac{300}{1 + 10e^{0.5x}}$

9. $f(x) = \frac{300}{2 + 10e^{0.5x}}$

Quick Review Solutions

Factor the expression and use sign charts to solve the inequality.

1. $x^2 - 4 < 0$ $(-2, 2)$

2. $x^3 - 9x > 0$ $(-3, 0) \cup (3, \infty)$

Find the domain of f and f' .

3. $f(x) = 2xe^x$ f : all reals; f' : all reals

4. $f(x) = x^{1/3}$ f : all reals; f' : $x \neq 0$

5. $f(x) = \frac{x}{x-1}$ f : $x \neq 1$; f' : $x \neq 1$

Quick Review Solutions

Find the horizontal asymptotes of the function's graph.

6. $f(x) = (9 - x^2) e^x$ $y=0$

7. $f(x) = (9 - x^2) e^{-x}$ $y=0$

8. $f(x) = \frac{300}{1 + 10e^{0.5x}}$ $y=0$ and $y=300$

9. $f(x) = \frac{300}{2 + 10e^{0.5x}}$ $y=0$ and $y=150$

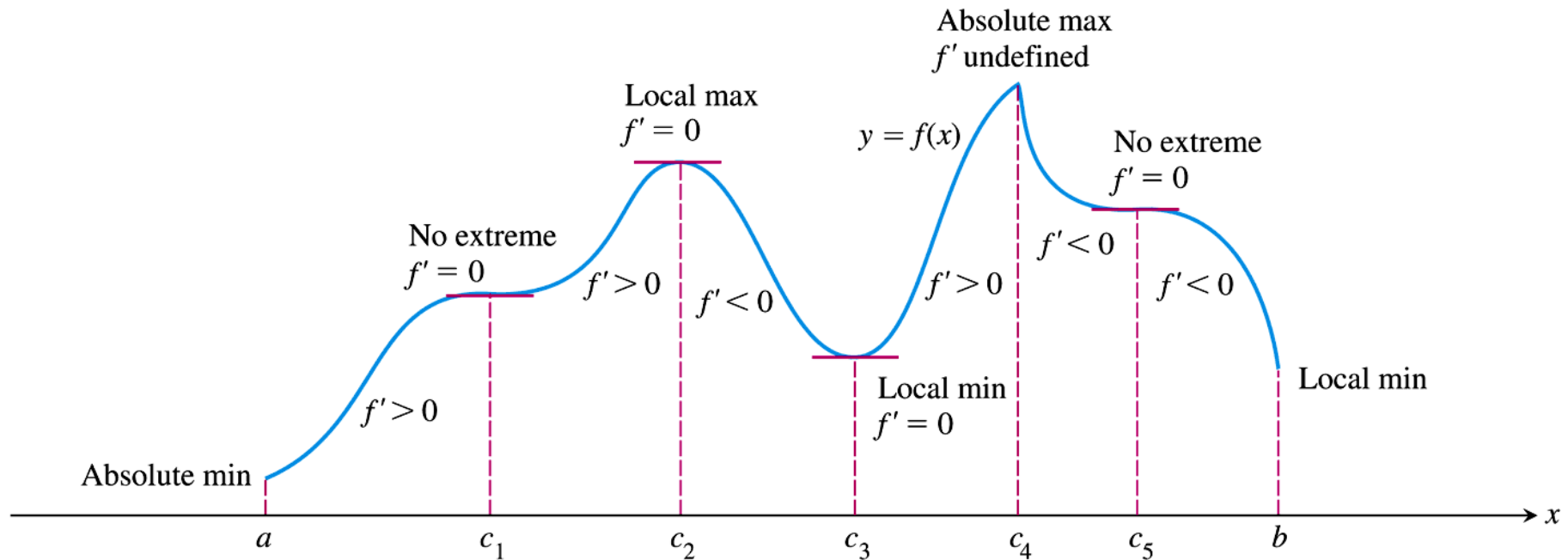
What you'll learn about

- Intervals of increase or decrease
- Both First and Second Derivative Tests for local extrema
- Intervals of upward or downward concavity
- Points of inflection
- Identification of key features of functions and their derivatives

...and why

Differential calculus is a powerful problem-solving Tool precisely because of its usefulness for analyzing functions.

First Derivative Test for Local Extrema



First Derivative Test for Local Extrema

The following test applies to a continuous function $f(x)$.

At a critical point c :

1. If f' changes sign from positive to negative at c , then f has a local maximum value at c .
2. If f' changes sign from negative to positive at c , then f has a local minimum value at c .
3. If f' does not change sign at c , then f has no local extreme value at c .

At a left endpoint a :

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a .

At a right endpoint b :

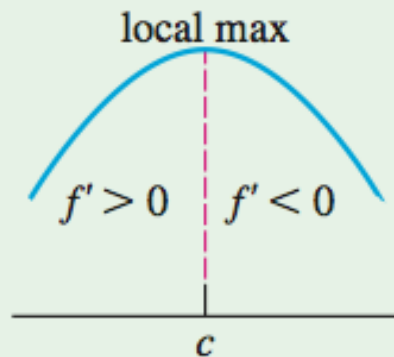
If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b .

First Derivative Test for Local Extrema

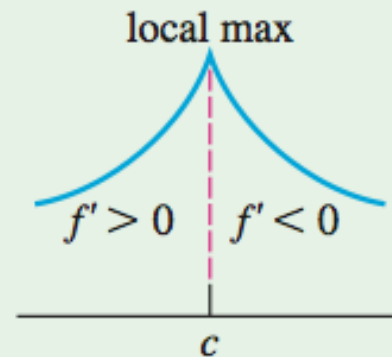
The following test applies to a continuous function $f(x)$.

At a critical point c :

1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c .



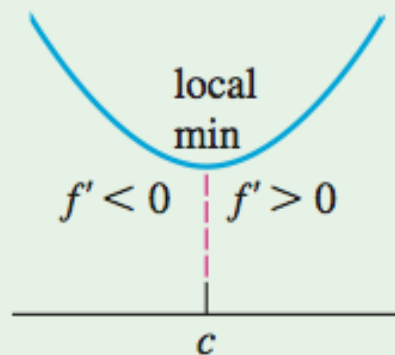
(a) $f'(c) = 0$



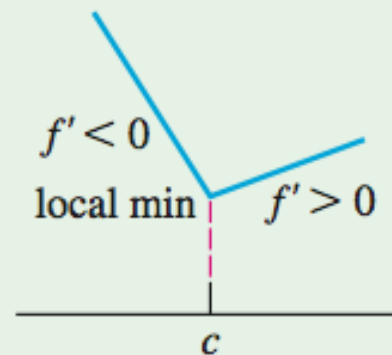
(b) $f'(c)$ undefined

First Derivative Test for Local Extrema

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$) then f has a local minimum value at c .



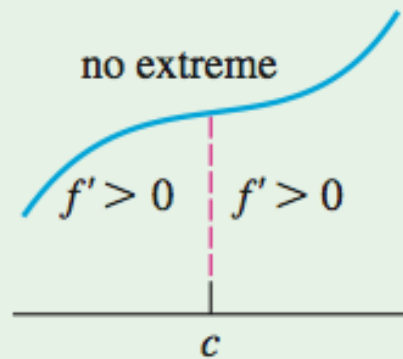
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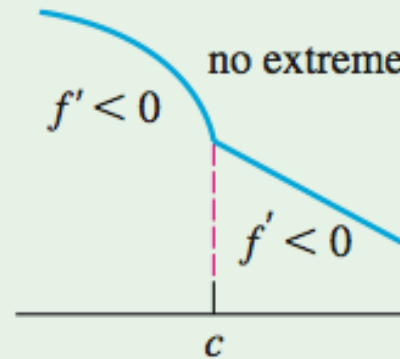
(b) $f'(c)$ undefined

First Derivative Test for Local Extrema

3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c .



(a) $f'(c) = 0$

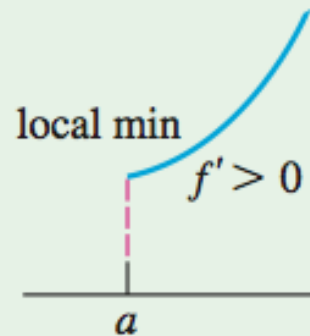
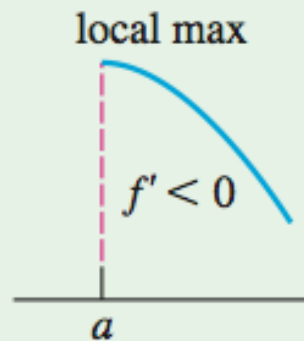


(b) $f'(c)$ undefined

First Derivative Test for Local Extrema

At a left endpoint a :

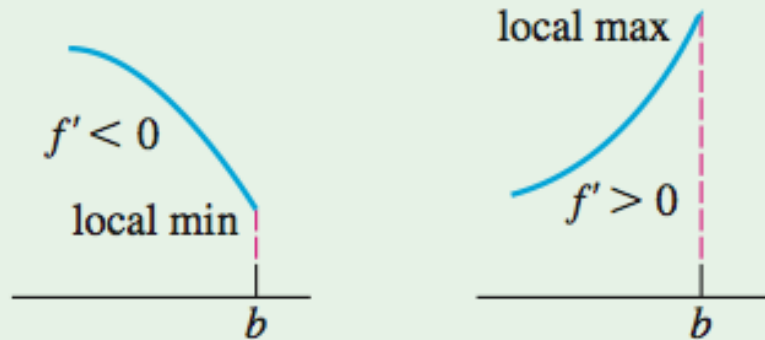
If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a .



First Derivative Test for Local Extrema

At a right endpoint b :

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b .



Example Using the First Derivative Test

Use the First Derivative Test to find the local extreme values. Identify any absolute extrema. $f(x) = x^3 - 27x + 3$

Since f is differentiable for all real numbers, the only critical points are the zeros of f' . Solving $f'(x) = 3x^2 - 27 = 0$, we find the zeros to be $x = -3$, and $x = 3$.

The zeros partition the x -axis into three intervals. Use a sign chart to find the sign on each interval. The First Derivative Test and the sign of f' tells us that there is a local maximum at $x = -3$ and a local minimum at $x = 3$. The local maximum value is $f(-3) = 57$ and the local minimum value is $f(3) = -51$.

The range of $f(x)$ is $(-\infty, \infty)$ so there is no absolute extrema.

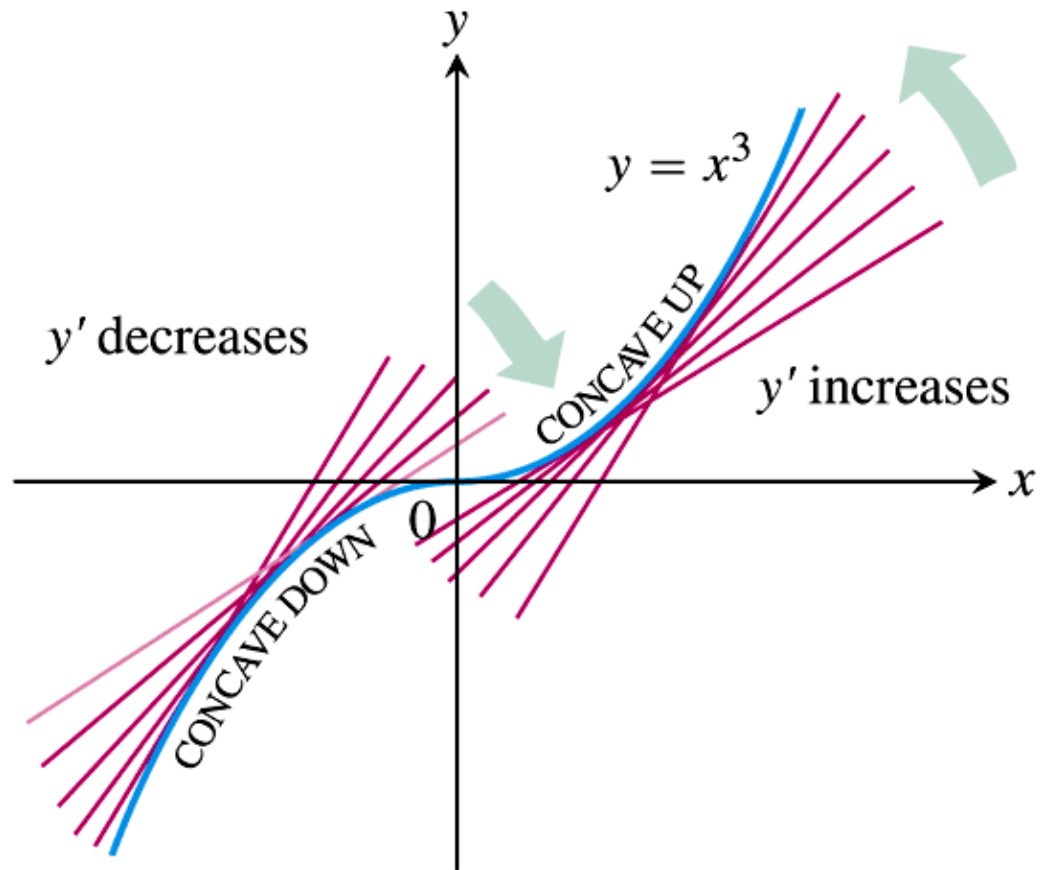
Concavity

The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval I if y' is increasing on I .

(b) concave down on an open interval I if y' is decreasing on I .

Concavity



Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on an open interval where $y'' > 0$.

(b) concave down on an open interval where $y'' < 0$.

Example Determining Concavity

Use the Concavity Test to determine the concavity of $f(x) = x^2$ on the interval $(2, 8)$.

Since $y'' = 2$ is always positive, the graph of $y = x^2$ is concave up on any interval. In particular, it is concave up on $(2, 8)$.

Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Example Finding Points of Inflection

Find all points of inflection of the graph of $y = 2e^{-x^2}$.

Find the second derivative of $y = 2e^{-x^2}$.

$$y' = 2e^{-x^2}(-2x) = -4xe^{-x^2}$$

$$\begin{aligned} y'' &= -4e^{-x^2} + (-4x)e^{-x^2}(-2x) \\ &= -4e^{-x^2} + (8x^2)e^{-x^2} \\ &= 4e^{-x^2}(-1 + 2x^2) \end{aligned}$$

The factor $4e^{-x^2}$ is always positive. The factor $(-1 + 2x^2)$ changes sign

at $x = \pm\sqrt{\frac{1}{2}}$. The points of inflection are $\left(-\sqrt{\frac{1}{2}}, \frac{2}{\sqrt{e}}\right)$ and $\left(\sqrt{\frac{1}{2}}, \frac{2}{\sqrt{e}}\right)$.

Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Example Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 6x + 5$.

$$f'(x) = 3x^2 - 6$$

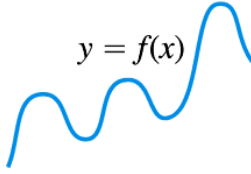
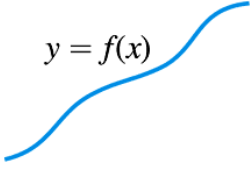
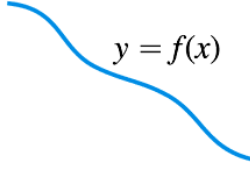
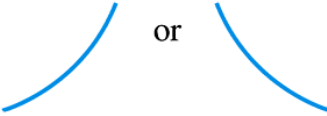
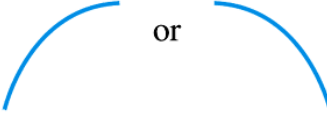
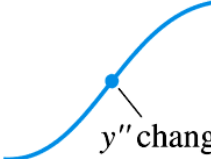
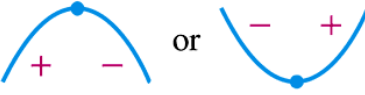
$$f''(x) = 6x$$

Test the critical points $x = \pm\sqrt{2}$.

$f''(-\sqrt{2}) = -6\sqrt{2} < 0 \Rightarrow$ has a local maximum at $x = -\sqrt{2}$ and

$f''(\sqrt{2}) = 6\sqrt{2} > 0 \Rightarrow$ has a local minimum at $x = \sqrt{2}$.

Learning about Functions from Derivatives

 $y = f(x)$ Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y = f(x)$ $y' > 0 \Rightarrow$ graph rises from left to right; may be wavy	 $y = f(x)$ $y' < 0 \Rightarrow$ graph falls from left to right; may be wavy
 or $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 or $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 y'' changes sign Inflection point
 or y' changes sign $\Rightarrow$ graph has local maximum or minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Quick Quiz for Sections 5.1 – 5.3

You should solve these problems without using a graphing calculator.

1. How many critical points does the function $f(x) = (x - 2)^2 (x + 3)^4$ have?

- (A) One
- (B) Two
- (C) Three
- (D) Five
- (E) Nine

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Quick Quiz for Sections 5.1 – 5.3

2. For what value of x does the function $f(x) = (x - 2)(x - 3)^2$ have a relative maximum?

- (A) - 3
- (B) - 7/3
- (C) - 5/2
- (D) 7/3
- (E) 5/2

Quick Quiz for Sections 5.1 – 5.3

2. For what value of x does the function $f(x) = (x - 2)(x - 3)^2$ have a relative maximum?

(A) - 3

(B) - 7/3

(C) - 5/2

(D) 7/3

(E) 5/2

Quick Quiz for Sections 5.1 – 5.3

3. If g is a differentiable function such that $g(x) < 0$ for all real numbers x , and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

- (A) f has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.
- (B) f has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.
- (C) f has a relative minima at $x = -3$ and at $x = 3$.
- (D) f has a relative maxima at $x = -3$ and at $x = 3$.
- (E) It cannot be determined if f has any relative extrema.

Quick Quiz for Sections 5.1 – 5.3

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(E) It cannot be determined if f has any relative extrema.